

Exercise 17

In Exercises 15-20, you are asked to solve the wave equation (1) subject to the boundary conditions (3), (4), and the initial conditions (5), (6), for the given data. [Hint: Use (10) and the remark that follows it.]

$$f(x) = \frac{1}{2} \sin \frac{\pi x}{L} + \frac{1}{4} \sin \frac{3\pi x}{L} + \frac{2}{5} \sin \frac{7\pi x}{L}, \quad g(x) = 0$$

Solution

The general solution to the wave equation on a finite interval with fixed ends and arbitrary initial shape and zero velocity,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad -\infty < t < \infty$$

$$u(x, 0) = \frac{1}{2} \sin \frac{\pi x}{L} + \frac{1}{4} \sin \frac{3\pi x}{L} + \frac{2}{5} \sin \frac{7\pi x}{L}$$

$$\frac{\partial u}{\partial t}(x, 0) = 0$$

$$u(0, t) = 0$$

$$u(L, t) = 0,$$

is (to be derived in later chapters)

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}.$$

To determine the constants A_n , set $t = 0$ and substitute the given function for $u(x, 0)$.

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

$$\frac{1}{2} \sin \frac{\pi x}{L} + \frac{1}{4} \sin \frac{3\pi x}{L} + \frac{2}{5} \sin \frac{7\pi x}{L} = A_1 \sin \frac{\pi x}{L} + A_2 \sin \frac{2\pi x}{L} + A_3 \sin \frac{3\pi x}{L} + \cdots$$

Then match the coefficients on both sides.

$$A_1 = \frac{1}{2}$$

$$A_3 = \frac{1}{4}$$

$$A_7 = \frac{2}{5}$$

⋮

$$A_n = 0, \quad n \neq 1, 3, 7$$

Therefore, the general solution that satisfies the initial conditions is

$$\begin{aligned} u(x, t) &= A_1 \sin \frac{\pi x}{L} \cos \frac{\pi ct}{L} + A_3 \sin \frac{3\pi x}{L} \cos \frac{3\pi ct}{L} + A_7 \sin \frac{7\pi x}{L} \cos \frac{7\pi ct}{L} \\ &= \frac{1}{2} \sin \frac{\pi x}{L} \cos \frac{\pi ct}{L} + \frac{1}{4} \sin \frac{3\pi x}{L} \cos \frac{3\pi ct}{L} + \frac{2}{5} \sin \frac{7\pi x}{L} \cos \frac{7\pi ct}{L}. \end{aligned}$$